ON CONSTRUCTION OF GROUP DIVISIBLE AND RECTANGULAR DESIGNS

By

A. Dey, C K. Midha and H. T. Trivedi

Institute of Agricultural Research Statistics, New Delhi-12

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1. Introduction

Group Divisible (GD) designs form an important class of two-associate PBIB designs. Several methods of construction of these designs are available in literature. GD designs have been extensively tabulated by Bose, Clatworthy and Shrikhande (1954) and subsequently by Clatworthy (1956). In section 2, we describe two methods of construction of these designs.

The Rectangular association scheme, introduced by Vartak (1955) is an important three class association scheme. PBIB designs based on the Rectangular scheme may be called Rectangular designs. In section 3, we present a method of construction of these designs, using two Balanced Incomplete Block (BIB) designs. It is also shown that if one of these BIB designs belongs to the Family (A), characterised by $b=4(r-\lambda)$, then the resultant design is GD.

2. Construction of GD Designs

Method 1 : For the construction of GD designs, we make use of Orthogonal arrays of strength two.

A $v' \times N'$ matrix A with entries from a set Σ of $s \geqslant 2$ elements is called an Orthogonal Array of strength t, size N', v' constraints and s symbols, if each $t \times N'$ submatrix of A contains all possible $t \times 1$ column vectors with the same frequency μ. The integer μ is called the index of the array, and the array is denoted by (N', v', s, t, μ). For our purpose, we shall use the arrays $(N', v', 2, 2, \mu)$ with symbols 0 and 1 (without loss of generality).

Let N^* be the usual incidence matrix of a BIB design with parameters v'', b'', r'', k'', λ'' such that $v''=2k''$. Let $\bar{N}^*$ be the incidence matrix of the complementary BIB design of N^* and let there exist an orthogonal array A with parameters $(N', v', 2, 2, \mu)$. We replace the unity in A by N^* and zero by $\bar{N}^*$ and call the derived matrix as N. Then, we have the following :

Theorem 1. The matrix N is the incidence matrix of a Semi-regular GD design with the following parameters :—

$$v=v'v'', b=N'b'', r=N'b''/2, k=v'k'', \lambda_1=N'\lambda'',$$
$$\lambda_2=\mu b'', m=v', n=v''.$$

where the symbols $v, b, \ldots$, etc., have their usual significance.

Proof: The expression for v, b, r, and k are obvious. Now, let the rows of the matrix N be numbered by the ordered pair (i, j), $i=1, 2, \ldots, v'$, $j=1, 2, \ldots, v''$. We call any two symbols (α, β) and (α', β') as first associates if $\alpha=\alpha'$; otherwise they are second associates. Obviously, this association rule is the GD association scheme with $m=v'$, $n=v''$. For calculating the λ-parameters, we proceed as follows :

The contribution towards λ_1 will come from any row of A. In every row of A, there are an equal number of unities and zeros and since the unity in A is replaced by N^*, and zero by N^* λ_1 is given by

$$\lambda_1=(N'/2) \text{ (inner product of any two rows of } N^*)$$
$$+(N'/2) \text{ (inner product of any two rows of } \bar{N}^*)$$

or $$\lambda_1=N'\lambda''/2+N'(b''-2r''+\lambda'')/2$$
$$=N'\lambda'', \text{ for, } b''=2r''.$$

Proceeding on similar lines, we can show that $\lambda_2=\mu b''$.

Hence the theorem.

Method 2 : A square matrix H of order N with entries ± 1 is said to be Hadamard if $H'H=HH'=NI_N$. A Hadamard matrix is said to be 'normalised' if its first row and first column contains only $+1$'s. We shall use the normalised Hadamard matrices for the construction of GD designs.

Let H be a normalised Hadamard matrix of order $N=4t$, and let N^* and $\bar{N}^*$ be as defined earlier. In H, replace the unity by N^* and -1 by $\bar{N}^*$ and called the derived matrix as N. Then, one can easily prove the following result.

Theorem 2. The matrix N is the incidence matrix of a semi-regular GD design with parameters

$$v=4t\,v'', b=4t\,b'', r=4t\,r'', k=4t\,k'',$$
$$\lambda_1=4t\,\lambda'', \lambda_2=2t\,r'', m=4t, n=v''.$$

3. Construction of Rectangular Design

3.1. The Rectangular association scheme was defined by Vartak (1955) as follows :

There are $v=mn$ symbols (treatments) arranged in an $m \times n$ array. With reference to any symbol θ, the first associates are precisely those which occur in the same row of the array with θ, second associates are those which occur in the same column of the array and rest are third associates. Alternatively one may define the scheme in the following manner :—

Let the mn symbols be indexed by the ordered pair (i, j), $i=1, 2, \ldots, m, j=1, 2, \ldots, n$. Any two symbols (α, β) and (α', β') are

first associates if $\alpha=\alpha'$, $\beta \neq \beta'$

second associates if $\alpha \neq \alpha'$, $\beta=\beta'$

and third associates if $\alpha \neq \alpha'$, $\beta \neq \beta'$.

The parameters of the association scheme are given in terms of m and n by Vartak (1955). We now proceed to describe a method of construction of PBIB designs based on the Rectangular association scheme.

Let there exist two BIB designs D_1 and D_2 with respective parameters v', b', r', k', λ', and v'', b'', r'', k'', λ''. Let N_1 and N_2 be the respective incidence matrices of D_1 and D_2, and $\bar{N}_2$ that of the complementary design of D_2. In N_1, replace the unity by N_2 and zero by $\bar{N}_2$ and call the derived matrix N. Then,

Theorem 3. The matrix N is the incidence matrix of Rectangular PBIB design D with the following parameters :—

$$v=v'v'',\ b=b'b'',\ r=r'r''+(b'-r')(b''-r''),$$
$$k=k'k''+(v'-k')(v''-k'')$$
$$\lambda_1=r'\lambda''+(b'-r')(b''-2r''+\lambda'')$$
$$\lambda_2=r''\lambda'+(b''-r'')(b'-2r'+\lambda')$$
$$\lambda_3=\lambda'\lambda''+(b'+2r'+\lambda')(b''-2r''+\lambda'')+2(r'-\lambda')(r''-\lambda''),$$
$$m=v',\ n=v''.$$

Proof : The expressions for the parameters v, b, r and k need no proof. Number the rows of N by the ordered pair (α, β) and define two symbols (α, β) and (α', β') $(\alpha, \alpha'=1, 2, \ldots, v';\ \beta, \beta'=1, 2, \ldots, v'')$ as first associates if $\alpha=\alpha'$, $\beta \neq \beta'$; second associates if $\alpha \neq \alpha'$, $\beta=\beta'$ and third associates otherwise. Obviously this rule is the Rectangular association scheme with $m=v'$, $n=v''$. We need now show only the value of the λ-parameters.

Consider any row of N. In this row, N_2 occurs r' times and $\bar{N}_2$, $(b'-r')$ times. Noticing that the inner product of any two rows of N_2 is λ'' and that of $\bar{N}_2$ is $b''-2r''+\lambda''$, we obviously have

$$\lambda_1=r'\lambda''+(b'-r')(b''-2r''+\lambda'').$$

Consider now any two row vectors of N, numbered say (i, j) and (i', j), $i\neq i'$. Since in any two-rowed submatrix of N_1, the four ordered column vectors

$$\binom{1}{1},\quad \binom{1}{0},\quad \binom{0}{1} \text{ and } \binom{0}{0}$$

occur respectively λ' times, $(r'-\lambda')$ times, $(r'-\lambda')$ times and $(b'-2r'+\lambda')$ times, λ_2 can be computed as

$$\lambda_2=\lambda'(\alpha_i.\alpha_i)+(r'-\lambda')(\alpha_i.\beta_i)+(r'-\lambda')(\alpha_i.\beta_i)+(b'-2r'-\lambda')(\beta_i.\beta_i),$$

where α_i is the *i-th* row of N_2, β_i, that of $\bar{N}_2$ and $(a.b)$ is the inner product of a and b. This gives

$$\lambda_2=\lambda' r''+(b'-2r'+\lambda')(b''-r'').$$

On similar lines, we can show that

$$\begin{aligned}\lambda_3 &= \lambda'(\alpha_i.\alpha_j)+(r'-\lambda')(\alpha_i.\beta_j)+(r'-\lambda')(\beta_i.\alpha_j)+(b'-2r'+\lambda')(\beta_i.\beta_j)\\ &= \lambda'\lambda''+(b'-2r'+\lambda')(b''-2r''+\lambda'')+2(r'-\lambda')(r''-\lambda'').\end{aligned}$$

3.2. It is well known that the Rectangular association scheme can always be reduced to a Group Divisible associatton scheme by calling the second and the third associates of the Rectangular scheme, the second associates of the GD association scheme. We now investigate the condition under which the Rectangular design D obtained above reduces to a GD design.

We have for the design D,

$$\lambda_2=\lambda' r''+(b'-2r'+\lambda')(b''-r'')$$

and $\lambda_3=\lambda'\lambda''+(b'-2r'+\lambda')(b''-2r''+\lambda'')+2(r'-\lambda')(r''-\lambda'')$.

The design D will be a GD design if and only if $\lambda_2=\lambda_3$. It is easy to see that $\lambda_2=\lambda_3$ if and only if $b'=4(r'-\lambda')$, that is, *iff* D_1 belongs to the Family (A) of BIB designs. Thus,

Theorem 4. A necessary and sufficient condition for the design D to be Group Divisible is that D_1 belongs to the Family (A) of BIB designs.

As a corollary, we obtain the result of Shrikhande (1962).

Corollary : If D_1 and D_2 both belong to the family (A) of BIB designs, then D is a BIB design which also belongs to family (A).

SUMMARY

Two methods of construction of Group Divisible (GD) designs and a method of construction of Rectangular designs are discussed.

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